Lecture No. 16

Example

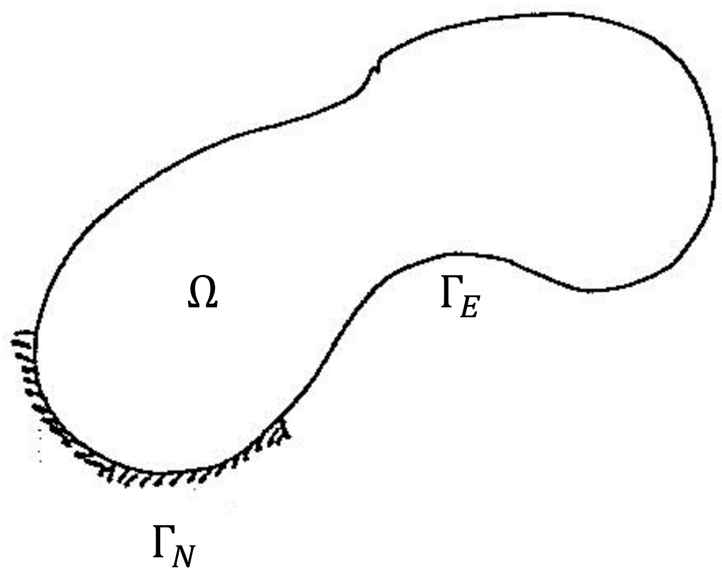
Consider Laplace's Equation

$$h_x \frac{\partial^2 u}{\partial x^2} + h_y \frac{\partial^2 u}{\partial y^2} = 0$$

with b.c.'s

$$u = \bar{u} \text{ on } \Gamma_E$$

$$q_n = \alpha_{nx} h_x \frac{\partial u}{\partial x} + \alpha_{ny} h_y \frac{\partial u}{\partial y} = \bar{q}_n \text{ on } \Gamma_E$$



- Recall that α_{nx} and α_{ny} are the direction cosines of the normal *n*, w.r.t. *x* and *y*
- Assuming Galerkin, the fundamental weak form is:

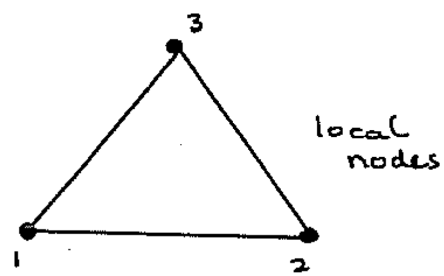
$$\iint_{\Omega} \left(h_x \frac{\partial u}{\partial x} \frac{\partial(\delta u)}{\partial x} + h_y \frac{\partial u}{\partial y} \frac{\partial(\delta u)}{\partial y} \right) dxdy = \int_{\Gamma_N} \bar{q}_n \delta u d\Gamma$$

We assumed that h_x and h_y are constants.

- We now establish the approximation over the element. We will use triangular elements.

Using linear interpolation over the triangle we have:

$$u = \underline{\phi}\,\underline{u}^{(n)} = [\xi_1 \quad \xi_2 \quad \xi_3] \begin{bmatrix} u_1^{(n)} \\ u_2^{(n)} \\ u_3^{(n)} \end{bmatrix} = \xi_1 u_1^{(n)} + \xi_2 u_2^{(n)} + \xi_3 u_3^{(n)}$$



- Substituting into our formulation we have:

$$\sum_{el} \iint_{\Omega^{(n)}} \left(h_x \underline{\phi}_{,x} \underline{u}^{(n)} \underline{\phi}_{,x} \delta \underline{u}^{(n)} + h_y \underline{\phi}_{,y} \underline{u}^{(n)} \underline{\phi}_{,y} \delta \underline{u}^{(n)} \right) dxdy - \int_{\Gamma_N} \bar{q}_n \underline{\phi}\, \delta \underline{u}^{(n)} d\Gamma = 0$$

$$\Rightarrow$$

$$\sum_{el} \delta \underline{u}^{(n)^T} \left\{ \left[h_x \int \int_{\Omega^{(n)}} \underline{\phi}_{,x}^T \underline{\phi}_{,x} d\Omega \right] \underline{u}^{(n)} + \left[h_y \int \int_{\Omega^{(n)}} \underline{\phi}_{,y}^T \underline{\phi}_{,y} d\Omega \right] \underline{u}^{(n)} - \left[\int_{\Gamma_N} \bar{q}_n \underline{\phi}^T d\Gamma \right] \right\} = 0$$

$$\Rightarrow$$

$$\sum_{el} \delta \underline{u}^{(n)^T} \left\{ \underline{S}^{(n)} \underline{u}^{(n)} - \underline{P}^{(n)} \right\} = 0$$

where

$$\underline{S}^{(n)} = h_x \int \int_{\Omega^{(n)}} \underline{\phi}_{,x}^T \underline{\phi}_{,x} d\Omega + h_y \int \int_{\Omega^{(n)}} \underline{\phi}_{,y}^T \underline{\phi}_{,y} d\Omega$$

$$\underline{P}^{(n)} = \int_{\Gamma_N} \bar{q}_n \underline{\phi}^T d\Gamma$$

Summing, taking into account functional continuity and the arbitrary variation of $\delta\underline{u}$, we get the following system of global equations:

$$\underline{\delta u} = \underline{P}$$

However we recall that

$$\underline{\phi}_{,x} = \frac{1}{2A}[b_1 \quad b_2 \quad b_3] = \underline{b}$$

$$\underline{\phi}_{,y} = \frac{1}{2A}[a_1 \quad a_2 \quad a_3] = \underline{a}$$

where a_1 and b_1 relate to arithmetic differences in nodal coordinates. Thus $\underline{a}$ and $\underline{b}$ are constant vectors and:

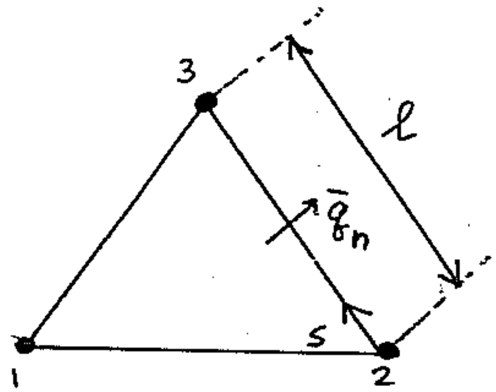
$$\underline{S}^{(n)} = h_x \int \int_{\Omega^{(n)}} \underline{b}^T \underline{b} d\Omega + h_y \int \int_{\Omega^{(n)}} \underline{a}^T \underline{a} d\Omega$$

$$\Rightarrow$$

$$\underline{S}^{(n)} = h_x \underline{b}^T \underline{b} A + h_y \underline{a}^T \underline{a} A$$

- How do we handle natural boundary terms? Assume segment 2-3 of an element is on the natural boundary. Then:

$$\underline{P}^{(n)} = \int_{node2}^{node3} \underline{\phi}^T \underline{\bar{q}}_n \, d\Gamma$$



Let's assume that $\bar{q}_n$ is constant over the segment (i.e. it could vary in a prescribed polynomial or other manner):

$$\underline{P}^{(n)} = \bar{q}_n \int_{node2}^{node3} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} dS$$

However on side 1, $\xi_1 = 0$ and:

$$\underline{P}^{(n)} = \bar{q}_n \int_{node2}^{node3} \begin{bmatrix} 0 \\ \xi_2 \\ \xi_3 \end{bmatrix} dS$$

Using the analytical integration formula.

$$\int \xi_2 dS = \frac{1!\,0!}{(1+0+1)} L = \frac{1}{2} l_{2-3}$$

Similarly:

$$\int \xi_3 dS = \frac{1}{2} l_{2-3}$$

Therefore:

$$\underline{P}^{(n)} = \bar{q}_n \begin{bmatrix} 0 \\ \frac{1}{2} l_{2-3} \\ \frac{1}{2} l_{2-3} \end{bmatrix}$$

<u>Mass Matrices</u>

- Let's examine a typical mass matrix (which is associated with the time derivative term of the 2-D C-D equation)

$$\int_{\Omega} u_{,t} \delta u d\Omega = \sum_{el} \int_{\Omega^{(n)}} \left(\underline{\phi}\,\underline{u}^{(n)}\right)_{,t} \underline{\phi}\,\delta \underline{u}^{(n)} d\Omega$$

$$\Rightarrow$$

$$\int_{\Omega} u_{,t} \delta u d\Omega = \sum_{el} \delta \underline{u}^{(n)^T} \left[\int_{\Omega^{(n)}} \underline{\phi}^T \underline{\phi} d\Omega \right] u_{,t}^{(n)}$$

Therefore

$$\underline{M}^{(n)} = \left[\int_{\Omega^{(n)}} \underline{\phi}^T \underline{\phi} d\Omega \right]$$

For a linear triangle we have $\underline{\phi} = [\xi_1 \xi_2 \xi_3]$. Thus

$$\underline{M}^{(n)} = \int \int_{\Omega^{(n)}} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} [\xi_1 \xi_2 \xi_3] d\Omega = \int \int_{\Omega^{(n)}} \begin{bmatrix} \xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\ \xi_2 \xi_1 & \xi_2^2 & \xi_2 \xi_3 \\ \xi_3 \xi_1 & \xi_3 \xi_2 & \xi_3^2 \end{bmatrix}$$

We note that:

$$\int \int_{\Omega^{(n)}} \xi_i \xi_j d\Omega = \begin{cases} \dfrac{A^{(n)}}{12} & i \neq j \\ \dfrac{A^{(n)}}{6} & i = j \end{cases}$$

Thus

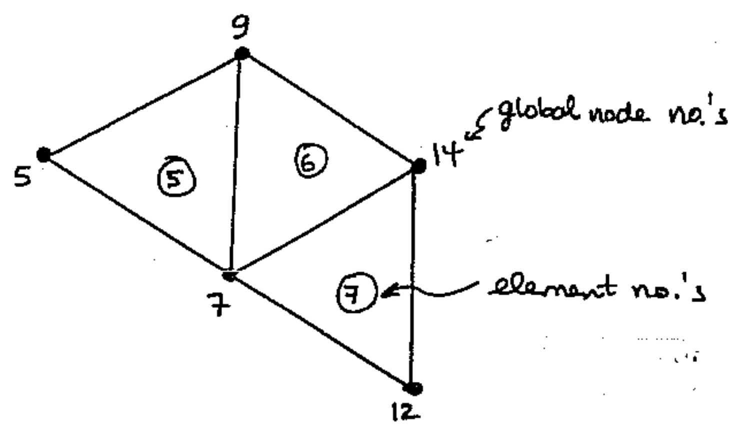
$$\underline{M}^{(n)} = \frac{A^{(n)}}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

If we were to lump this matrix we would have:

$$\underline{M}_L^{(n)} = \frac{A^{(n)}}{12} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \frac{A^{(n)}}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Structure of 2-D Global Mass Matrix</u>

Consider only a <u>portion</u> of a triangular grid with global node numbers and element numbers indicated:



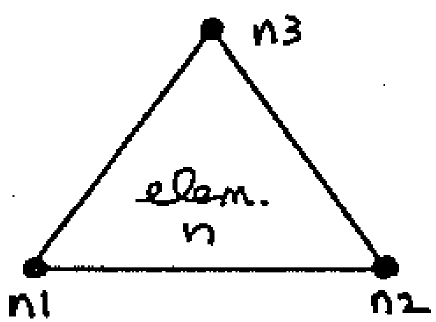
Procedure: Loop over the elements and sum local element matrices into "global" positions. Use element connectivity table to associate global with local node numbers for each element.

Element Connectivity Table:

element	n1	n2	n3
5	7	9	5
6	7	14	9
7	12	14	7

Indicate connectivity counter-clockwise (to get the correct sign convention for derivatives and areas).

- General element *n*:



$$\Rightarrow \qquad \underline{M}^{(n)} = \frac{A^{(n)}}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

- For element no. 5 we have

$$\underline{M}^{(5)} = \frac{A^{(5)}}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

global positions are 7, 9 and 5.

- For element no. 6 we have:

$$\underline{M}^{(6)} = \frac{A^{(6)}}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

global positions are 7, 14 and 9.

- For element no. 7 we have:

$$\underline{M}^{(7)} = \frac{A^{(7)}}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

global positions are 12, 14 and 7.

- We assemble these elemental matrices into the following global matrix:

$\left(\frac{1}{12}\right)$

global position	5	6	7	8	9	10	11	12	13	14
5	$2A^5$		A^5		A^5					
6										
7	A^5		$2A^5+2A^6+2A^7$		A^5+A^6			A^7		A^6+A^7
8										
9	A^5		A^5+A^6		$2A^6+2A^6$					A^6
10										
11										
12			A^7					$2A^7$		A^7
13										
14			A^6+A^7		A^6			A^7		$2A^6+2A^7$

zero elements outside of bandwidth

The contributions depend on element connectivity

If a node is part of 2 elements, it will make 2 contributions to a given location in the global matrix.

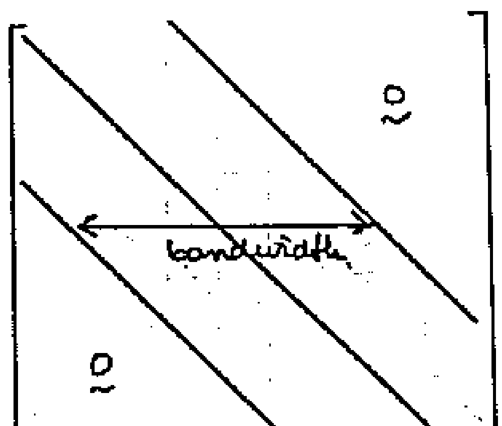
If a node is part of 3 elements, it will make 3 contributions to a given location in the global matrix.

- Bandwidth = 2 x maximum nodal point difference + 1

element	max. nodal pt. difference
5	$9-5=4$
6	$14-7=7$
7	$14-7=7$

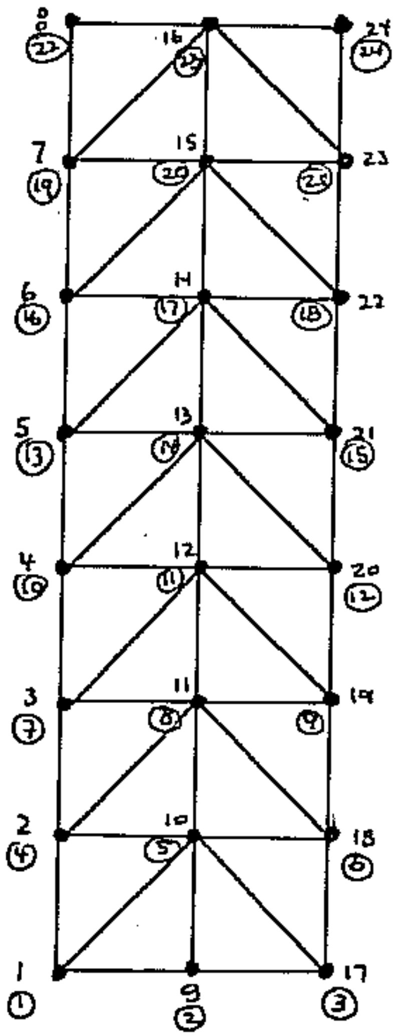
Total bandwidth = 2 x 7 + 1 = 15

Thus the band extends 7 to the right of the diagonal and 7 to the left of the diagonal (for equation 14). Therefore matrix structure is sparse and banded.



Bandwidth: maximum extent to within which nonzero locations are contained. Thus node numbering is extremely important to the efficiency of implementation of FE codes (assuming you're using banded matrix solvers).

- Example:



\# max nodal pt diff = 9

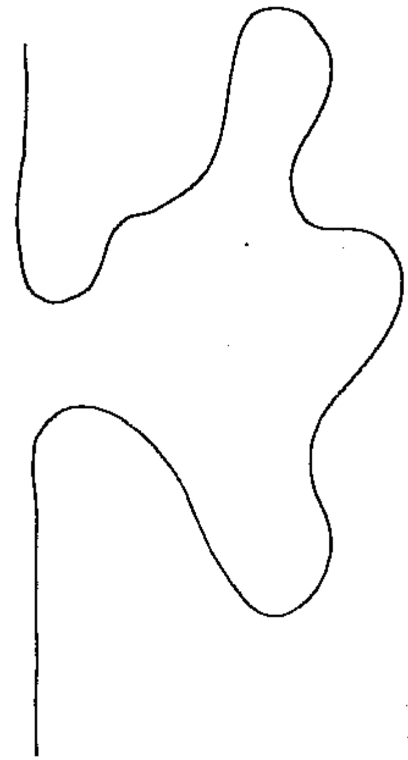
bandwidth = 19

\# max nodal pt diff = 4

bandwidth = 9

- Gauss solution procedures for banded matrices require $0(M^2N)$ operations where N equals the number of nodes, M equals the bandwidth. Therefor the savings of the example considered equal $0\left(\frac{19}{9}\right)^2 = 0(5)$. Savings can however be orders of magnitude!
- A rule of thumb for numbering is to number across the minimum width.
- For general and irregularly shaped geometries:

  Program algorithm to minimize maximum bandwidth by renumbering nodes. Optimal Node Number is typically not achieved but at least it's pretty good. (Also depends on the particular algorithm).



- Also we can use sparse matrix solvers. Node numbering can still be important for some iterative solvers such as pre-conditioned conjugate gradient solvers. We typically use these techniques for very large grids for which the efficiency in terms of CPU and memory utilization is much better than banded direct solvers.